One loop calculations in gauge theories regulated on an x^+-p^+ lattice

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In earlier work, the planar diagrams of $SU(N_c)$ gauge theory have been regulated on the light cone by a scheme involving both discrete p^+ and $\tau = ix^+$. The transverse coordinates remain continuous, but even so all diagrams are rendered finite by this procedure. In this scheme quartic interactions are represented as two cubics mediated by short-lived fictitious particles whose detailed behavior could be adjusted to retain properties of the continuum theory, at least at one loop. Here we use this setup to calculate the one loop three gauge boson triangle diagram, and so complete the calculation of diagrams renormalizing the coupling to one loop. In particular, we find that the cubic vertex is correctly renormalized once the couplings to the fictitious particles are chosen to keep the gauge bosons massless.

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I. INTRODUCTION

The large N_c limit of $SU(N_c)$ gauge theory, introduced long ago by 't Hooft [1], remains unsolved even though the limit singles out the planar Feynman diagrams of perturbation theory. Although a dramatic simplification, the sum of all planar diagrams still represents a rich enough dynamics to frustrate many imaginative approaches to their solution [2]. But even if an exact analytic solution is out of reach, this should not mean that the nonperturbative physics of the limit is hopelessly intractable. Indeed, in the face of the even richer dynamics of gauge theory at finite N_c, much insight has been gleaned by numerically studying lattice gauge theory [3–5] on a finite lattice as a useful nonperturbative model of the continuum gauge theory. Moreover, since the lattice could in principle be taken ever larger in size, lattice gauge theory can provide a concrete definition of continuum gauge theory.

Of course, one approach to the large N_c limit would be simply to study lattice gauge theory for ever larger N_c [6]. This is certainly a well-posed and interesting formulation of the problem. However, it is somewhat removed from the appealing and intuitive idea [7] that the large planar diagrams known as fishnets could, in a confining theory, provide a model of a QCD string "world-sheet." This idea is now even more compelling because of the conjectured equivalence of certain supersymmetric large N_c gauge theories to supergravity or superstring theories [8]. It would be nice to have an explicit discretized model representation of the sum of planar diagrams, which, if analytic methods fail, can at least be analyzed numerically.

Accordingly, Bering, Rozowsky, and Thorn (BRT) [9] reconsidered and refined earlier attempts [10–12] to construct a lattice model of the sum of planar diagrams by working in an infinite momentum frame (i.e. on a light front) and discretizing $\tau = ix^+$, imaginary light-cone time, and p^+, the kinematic light-cone momentum (p^- is the light-cone Hamiltonian). This x^+-p^+ lattice cuts off all ultraviolet and infra-red divergences because the discretization of τ and p^+ excludes zero values for these variables. The motivation for this approach is certainly not to simplify the calculation of individual Feynman diagrams: lattice calculations are rather more complicated than continuum ones [13–15], and considerably more complicated than those using the Mandelstam-Leibbrandt $i\epsilon$ prescription [16–18]. Rather the goal is to set up a systematic scheme which can be used to go beyond perturbation theory by summing diagrams. For example this setup is particularly helpful in identifying the sum of planar diagrams as a worldsheet system in the spirit of Bardakci and Thorn (BT) [19,20].

The purpose of this article is to further explore the viability of the BRT formalism as a regulator of perturbative diagrams and to deepen our understanding of how that formalism works. BRT calculated the one-loop gluon self-energy diagram, as a first check on the faithfulness of the x^+-p^+ lattice as a regulator of divergences, obtaining agreement with an earlier continuum calculation. However, to check asymptotic freedom, one still needs to calculate the one loop three gluon triangle in the same formalism. We therefore do the necessary additional one loop calculations in the pure $SU(N_c)$ gauge theory using the BRT discretization. In particular, we confirm that this scheme does not disturb the light-cone gauge asymptotic freedom calculations done earlier, using a "principal value" treatment of $p^+=0$ singularities Refs. [13–15]. We obtain expressions for the complete triangle diagram to one loop order and show that color charge is indeed correctly renormalized as in [15]. Further, we extract all of the divergences arising from the infrared region of small p^+. Although in individual diagrams there are double logarithms, we show that in the complete sum only single logarithms remain. It is an interesting highlight of the calculation that the double logarithms cancel between triangle and self-energy diagrams, term by term in the sum over the internal discretized p^+. In the context of the BT worldsheet formalism, this means that the cancellation occurs locally on the worldsheet, an encouraging outcome.

The Feynman rules used in [9] are unusual in several respects. First, the basic propagators are given in the mixed $ix^+, p^+, \mathbf{p}$ representation, with $ix^+ = ka$ and $p^+ = lm$, with $k,l = 1,2,3, \ldots$. The transverse polarization of the gluon is

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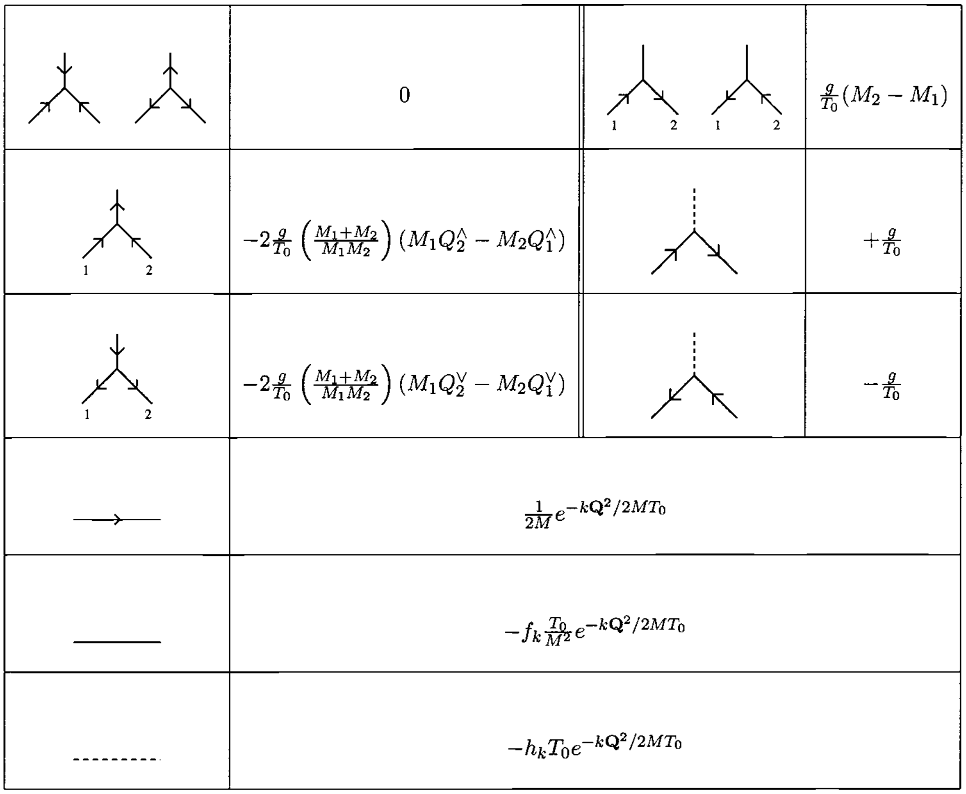

FIG. 1. Summary of discretized Feynman rules using only cubic vertices. We have explicitly inserted a factor of $a/m \equiv 1/T_0$ for each vertex arising from the discretization.

given in the complex basis $\wedge = (1+i2)/\sqrt{2}$, $\vee = (1-i2)/\sqrt{2}$, represented graphically by an arrow attached to the transverse gluon line. Next, since we focus on the planar diagrams of the $N_c \to \infty$ limit, there is no need for the double line notation, and all N_c dependence can be absorbed in the coupling constant $g \equiv g_s\sqrt{N_c}$. The resulting vertices for this restricted context accordingly depend on the cyclic ordering of the lines entering the vertex. Finally, all the quartic vertices, including those induced by integrating out A_+ in $A_- = 0$ gauge, are represented as the concatenation of two cubic vertices, with fictitious particles mediating the quartic interaction. The one mediating the induced quartic interaction (the instantaneous "Coulomb" exchange) can be thought of as a remnant of the A_+ field, whereas that mediating the $\mathrm{Tr}[A_k, A_l]^2$ vertex can be thought of as a remnant of F_{kl} in the first order form of the action. These fictitious particles would not propagate in the continuous time $a \to 0$ limit, but with a finite are allowed to propagate a limited number of time steps. This is implemented by including a k-dependent factor f_k or h_k in the propagator for each fictitious particle. These must satisfy $\Sigma_k f_k = \Sigma_k h_k = 1$ and must fall off rapidly with k, in order that the correct tree amplitudes be correctly produced. The Lorentz invariance of perturbation theory puts further constraints on their k dependence. Indeed, the flexibility offered by tuning these coefficients may even obviate the need for further explicit counterterms to guarantee Lorentz invariance. One constraint has already been put on this behavior in [9] by requiring the gauge boson to remain massless to one loop. We shall find that with no further constraints, color charge is correctly renormalized. The final set of Feynman rules for the transverse gauge bosons and the fictitious scalars in the context of planar diagrams is summarized in Fig. 1.

In this article we shall use this formalism exclusively. In Sec. II we illustrate how the cubic vertices with fictitious particles reproduce tree diagrams by calculating a four gauge boson amplitude, and by showing how the $p^+ = 0$ divergences are resolved in the tree approximation. Next we turn to the triangle diagram to one loop order. In Secs. III and IV we calculate the triangle diagram for three off-shell external transverse gluons. We extract all of the divergent parts, showing how the double logarithms arising from the entanglement of ultraviolet and infra-red divergences in light-cone gauge cancel in physical quantities.[1] Concluding remarks are given in Sec. V.

II. FOUR GLUON TREE DIAGRAMS

Because of the unusual nature of the Feynman rules in our discretized light-cone gauge, we introduce the reader to this formalism by discussing the tree approximation to the two gluon scattering diagrams shown in Fig. 2. We label the external momenta entering a diagram counterclockwise starting from the lower left and denote by k the discretized time difference between the vertices. Note that $p_i^+ = mM_i$. The t-channel exchange amplitudes for two up gluons to two up

[1] As stated earlier, the Mandelstam-Leibbrandt (ML) trick to avoid this entanglement is *not* employed in the setup used here. Indeed, one point we wish to stress is that, although the ML prescription is convenient for certain purposes, it is by no means necessary in a consistent formulation of perturbation theory.

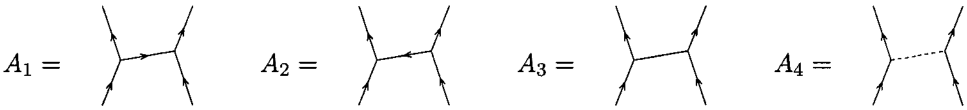

FIG. 2. The four gluon tree diagrams with t channel gluon exchange considered in this section.

gluons associated with these diagrams are then given by

$$\begin{aligned}
A_1 &= \frac{4g^2}{T_0^2}\frac{-M_1M_3}{2M_2M_4|M_1-M_4|^3} \\
&\quad\times\frac{(M_3p_2^{\wedge}-M_2p_3^{\wedge})(M_1p_4^{\vee}-M_4p_1^{\vee})}{e^{(p_1-p_4)^2/2|M_1-M_4|T_0}-1} \\
A_2 &= \frac{4g^2}{T_0^2}\frac{-M_2M_4}{2M_1M_3|M_1-M_4|^3} \\
&\quad\times\frac{(M_3p_2^{\vee}-M_2p_3^{\vee})(M_1p_4^{\wedge}-M_4p_1^{\wedge})}{e^{(p_1-p_4)^2/2|M_1-M_4|T_0}-1} \\
A_3 &= \frac{g^2}{T_0}\sum_{k=1}^{\infty} f_k e^{-k(p_1-p_4)^2/2|M_1-M_4|T_0} \\
&\quad\times\frac{(M_1+M_4)(M_2+M_3)}{(M_1-M_4)^2} \\
A_4 &= \frac{g^2}{T_0}\sum_{k=1}^{\infty} h_k e^{-k(p_1-p_4)^2/2|M_1-M_4|T_0}.
\end{aligned} \tag{2.1}$$

Note that we have supplied a wave function factor $e^{akp_i^+}$ ($e^{-akp_i^+}$) for each outgoing (incoming) particle, and we restrict the energies by the conservation law $p_3^+ + p_4^+ = p_1^+ + p_2^+$. Also, the absolute values account for the different treatment of the case $M_1<M_4$ from the case $M_1>M_4$. These expressions illustrate the essential features of the BRT discretization. A_1 and A_2 show the t channel poles due to one gluon exchange:

$$\frac{1}{e^{(p_1-p_4)^2/2|M_1-M_4|T_0}-1}\sim\frac{2|M_1-M_4|T_0}{(p_1-p_4)^2}. \tag{2.2}$$

Notice that time discretization has cut off the large $-t=(p_1-p_4)^2$ behavior of the amplitude exponentially. Recalling that $T_0=m/a$, we see that sending $a\to 0$ at fixed t and fixed $m|M_1-M_4|$ just leads to the usual $1/t$ factor of the Feynman tree amplitude. Notice also the role of p^+ discretization in cutting off $p^+=0$ singularities. In particular, with both $a,m\neq 0$ and $t<0$ the amplitudes are strictly zero for $M_1=M_4$, whereas after the continuum limit they are infinite at this point. With T_0 fixed, we can also reach the continuum limit by taking all $M_i\to\infty$ as well as $|M_1-M_4|\to\infty$. Either way, it is important to keep in mind that the correct $p_1^+\to p_4^+$ behavior of the continuum limit is only obtained if the continuum limit is first taken with $p_1^+\neq p_4^+$.

The expressions for A_3 and A_4 contain the so far undetermined parameters f_k, h_k. Since they limit the range of the k summation, the formal continuum limit just discussed only involves them in the combinations $\Sigma_k f_k$, $\Sigma_k h_k$, both of which are constrained to be unity. Thus, in the continuum limit, the sum of all four diagrams, $A=A_1+A_2+A_3+A_4$ is given by

$$\begin{aligned}
A &= \frac{g^2}{T_0}\Bigg[-\frac{4p_1^+p_3^+}{p_2^+p_4^+(p_1^+-p_4^+)^2} \\
&\quad\times\frac{(p_3^+p_2^{\wedge}-p_2^+p_3^{\wedge})(p_1^+p_4^{\vee}-p_4^+p_1^{\vee})}{(p_1-p_4)^2}-\frac{4p_2^+p_4^+}{p_1^+p_3^+(p_1^+-p_4^+)^2} \\
&\quad\times\frac{(p_3^+p_2^{\vee}-p_2^+p_3^{\vee})(p_1^+p_4^{\wedge}-p_4^+p_1^{\wedge})}{(p_1-p_4)^2} \\
&\quad+\frac{(p_1^++p_4^+)(p_2^++p_3^+)}{(p_1^+-p_4^+)^2}+1\Bigg].
\end{aligned} \tag{2.3}$$

It is significant that in the continuum limit no absolute value signs are needed.

The individual terms contributing to the continuum limit A show quadratic singularities as $p_1^+\to p_4^+$, typical of the light-cone gauge. But in the sum, these singularities are softened off shell and disappear on shell. To show this, we note the following identities:

$$\begin{aligned}
&(p_3^+p_2^{\wedge}-p_2^+p_3^{\wedge})(p_1^+p_4^{\vee}-p_4^+p_1^{\vee}) \\
&\quad=\frac{1}{2}(p_3^+p_2-p_2^+p_3)\cdot(p_1^+p_4-p_4^+p_1) \\
&\qquad+\frac{1}{2}(p_1^+-p_4^+)[(p_1^+-p_4^+)(p_2^{\wedge}p_1^{\vee}-p_2^{\vee}p_1^{\wedge}) \\
&\qquad-(p_1-p_4)^{\vee}(p_1^+p_2^{\wedge}-p_2^+p_1^{\wedge}) \\
&\qquad+(p_1-p_4)^{\wedge}(p_1^+p_2^{\vee}-p_2^+p_1^{\vee})]
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&(p_3^+p_2^{\vee}-p_2^+p_3^{\vee})(p_1^+p_4^{\wedge}-p_4^+p_1^{\wedge}) \\
&\quad=\frac{1}{2}(p_3^+p_2-p_2^+p_3)\cdot(p_1^+p_4-p_4^+p_1) \\
&\qquad-\frac{1}{2}(p_1^+-p_4^+)[(p_1^+-p_4^+)(p_2^{\wedge}p_1^{\vee}-p_2^{\vee}p_1^{\wedge}) \\
&\qquad-(p_1-p_4)^{\vee}(p_1^+p_2^{\wedge}-p_2^+p_1^{\wedge}) \\
&\qquad+(p_1-p_4)^{\wedge}(p_1^+p_2^{\vee}-p_2^+p_1^{\vee})].
\end{aligned} \tag{2.5}$$

Furthermore, by using energy momentum conservation we can rewrite the common first term on the right-hand side (rhs) of these identities:

$$
\begin{aligned}
&(p_3^+ p_2 - p_2^+ p_3)\cdot(p_1^+ p_4 - p_4^+ p_1)\\
&\quad = \frac{p_1^+ p_3^+ + p_2^+ p_4^+}{2}(p_1-p_4)^2 + \frac{(p_1^+ - p_4^+)^2}{2}(p_1+p_2)^2 + \frac{p_1^+ - p_4^+}{2}[p_2^+ p_4^2 + p_4^+ p_2^2 - p_1^+ p_3^2 - p_3^+ p_1^2].
\end{aligned}
\tag{2.6}
$$

The contribution of Eq. (2.6) to the continuum limit of A_1+A_2 is

$$
\begin{aligned}
(A_1+A_2)_{\mathrm{I}} = -\frac{g^2}{T_0}\Bigg[&\frac{(p_1^+ + p_4^+)(p_2^+ + p_3^+)}{(p_1^+ - p_4^+)^2} + 1 + \frac{(p_1^+ + p_2^+)^2(p_1^+ p_3^+ + p_2^+ p_4^+)}{p_1^+ p_2^+ p_3^+ p_4^+} + \frac{(p_1^{+2} p_3^{+2} + p_2^{+2} p_4^{+2})(p_1+p_2)^2}{p_1^+ p_3^+ p_2^+ p_4^+ (p_1-p_4)^2}\\
&+ \frac{(p_1^{+2} p_3^{+2} + p_2^{+2} p_4^{+2})[p_2^+ p_4^2 + p_4^+ p_2^2 - p_1^+ p_3^2 - p_3^+ p_1^2]}{p_1^+ p_3^+ p_2^+ p_4^+ (p_1^+ - p_4^+)(p_1-p_4)^2}\Bigg],
\end{aligned}
\tag{2.7}
$$

and we denote the rest of A_1+A_2 by $(A_1+A_2)_{\mathrm{II}}$. Thus, adding on A_3+A_4, we can write the total amplitude

$$
\begin{aligned}
A = (A_1+A_2)_{\mathrm{II}} - \frac{g^2}{T_0}\Bigg[&\frac{(p_1^+ + p_2^+)^2(p_1^+ p_3^+ + p_2^+ p_4^+)}{p_1^+ p_2^+ p_3^+ p_4^+} + \frac{(p_1^{+2} p_3^{+2} + p_2^{+2} p_4^{+2})(p_1+p_2)^2}{p_1^+ p_3^+ p_2^+ p_4^+ (p_1-p_4)^2}\\
&+ \frac{(p_1^{+2} p_3^{+2} + p_2^{+2} p_4^{+2})[p_2^+ p_4^2 + p_4^+ p_2^2 - p_1^+ p_3^2 - p_3^+ p_1^2]}{p_1^+ p_3^+ p_2^+ p_4^+ (p_1^+ - p_4^+)(p_1-p_4)^2}\Bigg].
\end{aligned}
\tag{2.8}
$$

We find that $(A_1+A_2)_{\mathrm{II}}$ has a continuum limit which has no $p_1^+ - p_4^+$ denominators:

$$
\begin{aligned}
(A_1+A_2)_{\mathrm{II}} = -\frac{2g^2}{T_0}\Bigg[&\frac{(p_1^+ p_3^+ + p_2^+ p_4^+)(p_1^+ + p_2^+)p_4^+(p_1^\wedge p_2^\vee - p_1^\vee p_2^\wedge)}{p_1^+ p_3^+ p_2^+ p_4^+ (p_1-p_4)^2}\\
&- \frac{(p_1^+ p_3^+ + p_2^+ p_4^+)(p_1^+ + p_2^+)[(p_2^+ p_1^\wedge - p_1^+ p_2^\wedge)p_4^\vee - (p_2^+ p_1^\vee - p_1^+ p_2^\vee)p_4^\wedge]}{p_1^+ p_3^+ p_2^+ p_4^+ (p_1-p_4)^2}\Bigg].
\end{aligned}
\tag{2.9}
$$

Note that the quadratically singular first term in square brackets on the rhs of Eq. (2.7) has been exactly canceled by the continuum limit of A_3, leaving only a linear singularity as $p_1^+ - p_4^+ \to 0$, which disappears when all of the external gluons are on shell ($p_i^2 = 0$):

$$
A_{\mathrm{div}} = \frac{(p_1^{+2} p_3^{+2} + p_2^{+2} p_4^{+2})[p_2^+ p_4^2 + p_4^+ p_2^2 - p_1^+ p_3^2 - p_3^+ p_1^2]}{p_1^+ p_3^+ p_2^+ p_4^+ (p_1^+ - p_4^+)(p_1-p_4)^2}.
\tag{2.10}
$$

This singular contribution has opposite signs for $p_1^+ > p_4^+$ and $p_1^+ < p_4^+$, so we can expect that, when this term occurs as a sub-diagram where $M_1 - M_4$ is summed, a principal value definition of the continuum integral approximation to the sum will be appropriate and no divergence will occur. It is also interesting to notice that when the particles in the initial and final state are equally off energy shell, i.e. $p_1^2/M_1 = p_2^2/M_2$ and $p_3^2/M_3 = p_4^2/M_4$, the quantity in square brackets becomes $(p_1^+ + p_2^+)(p_4^+ - p_1^+)(p_3^2/p_3^+ + p_1^2/p_1^+)$, with no singularity as $p_1^+ \to p_4^+$. The softening of $p^+ = 0$ singularities in this particular off-shell situation has also been noted in [21] in the radiative corrections to another four point (branion) process.

Finally, we comment that the on-shell four point amplitude assumes a fairly compact form in the Galilei center of mass frame $\mathbf{p}_2 = -\mathbf{p}_1$. In this case we obtain

$$
\begin{aligned}
A_{\text{on shell}}^{\mathrm{CM}} = -\frac{g^2}{T_0}\Bigg[&\frac{(p_1^{+2} p_3^{+2} + p_2^{+2} p_4^{+2})s}{p_1^+ p_3^+ p_2^+ p_4^+ t}\\
&+ \frac{(p_1^+ + p_2^+)^2(p_1^+ p_3^+ + p_2^+ p_4^+)}{p_1^+ p_2^+ p_3^+ p_4^+}\\
&\times\left(1 + 2\frac{p_1^\wedge p_4^\vee - p_1^\vee p_4^\wedge}{t}\right)\Bigg]
\end{aligned}
$$

where we have introduced the usual Mandelstam invariants $s = -(p_1+p_2)^2$ and $t = -(p_4-p_1)^2$. In this form it is easy to check that the diagram has the correct value.

For comparison, we record here the value of the diagram with s channel poles, also in the Galilei center of mass:

$$
A_s^{\mathrm{CM}} = -\frac{4g^2}{T_0}\frac{(p_1^+ + p_2^+)^4 p_1^\wedge p_4^\vee}{p_1^+ p_2^+ p_3^+ p_4^+ s}
\tag{2.11}
$$

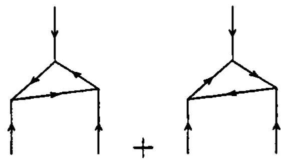

FIG. 3. Triangle with incoming arrows.

$$= -\frac{g^2}{T_0}\frac{(p_1^+ + p_2^+)^4}{p_1^+ p_2^+ p_3^+ p_4^+}\left[\frac{t}{s} + 2\frac{p_1^\wedge p_4^\vee - p_1^\vee p_4^\wedge}{s}\right] - \frac{g^2}{T_0}\frac{(p_1^+ + p_2^+)^2(p_1^+ p_3^+ + p_2^+ p_4^+)}{p_1^+ p_2^+ p_3^+ p_4^+}. \tag{2.12}$$

III. SIMPLE TRIANGLE WITH THREE TRANSVERSE GLUONS

We will now calculate the one loop three point function where all external gluons are transverse. We start with a particularly simple process in which the arrows on all external lines point inward. Since this process has no contribution at tree order, it cannot have ultraviolet divergences. In fact, we shall find that the diagram also has no infrared divergences and so is completely finite.

Using the discretized Feynman rules we see that only the two diagrams shown in Fig. 3 contribute to this choice of external momenta. One of these is shown in more detail in Fig. 4. The loop momentum is q and the momenta shown are routed along the arrows. This means that the momenta p_1, p_2 and p_3 are directed into the vertex and momentum conservation reads $\Sigma p_i = 0$. We use the convention that the discretized+component of momenta p_i is mM_i and that of q is ml. Accordingly we have $M_1, M_2 > 0$ and $M_3 = -M_1 - M_2 < 0$. For convenience we shall also use $M \equiv -M_3 > 0$ and $p \equiv -p_3 > 0$ in the following analysis. The discretized times are given by $\tau = ak$ and by translational invariance we can choose one of the vertices to be at $\tau = 0$. We then have a transverse loop integral to do, the integral over q^+ becomes a sum over l and finally we must sum over the two discretized times k_1 and k_2. When $k_1 > 0$ (so also $l > 0$), k_1 and $k_2' \equiv k_2 - k_1$ are independently summed from 1 to $+\infty$ and l is summed from 1 to $M_1 - 1$. When $k_1 < 0$ (so also $l < 0$), $k_1' \equiv -k_1$ and k_2 are independently summed from 1 to $+\infty$ and $l' \equiv -l$ is summed from 1 to $M_2 - 1$. By writing out the expression for the two diagrams, shifting the transverse loop

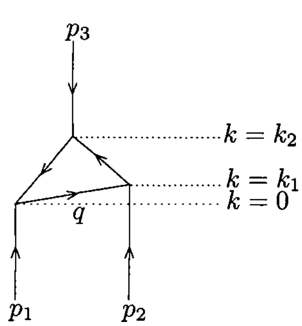

FIG. 4. Conventions when calculating the $\Gamma^{\wedge\wedge\wedge}$.

momentum and simplifying (note that a factor of e^{-akp^-} comes from each of the ingoing external lines), we obtain

$$\Gamma^{\wedge\wedge\wedge} = \sum_{k,l}\frac{g^3}{T_0^3}\frac{2(M_1M_2M_3)^{-1}}{(M_1 - l)(M_2 + l)|l|}e^{-H} \times \int\frac{d^2r}{(2\pi)^3}e^{-\Sigma T\mathbf{r}^2}\left(\frac{T_1}{\Sigma T}K_{23} - M_3 r\right)^\wedge \times\left(\frac{T_2}{\Sigma T}K_{31} - M_1 r\right)^\wedge\left(\frac{T_3}{\Sigma T}K_{12} - M_2 r\right)^\wedge$$

$$= \left(\frac{g}{T_0}\right)^3\sum_{k,l}\frac{e^{-H}}{4\pi^2}\frac{(M_1M_2M_3)^{-1}}{(M_1 - l)(M_2 + l)|l|} \times\frac{T_1T_2T_3}{\Sigma T^4}(K_{12}^\wedge)^3. \tag{3.1}$$

The sum on k,l is constrained as described above and for brevity we have made the following definitions:

$$T_1 = \frac{1}{2T_0}\frac{k_1}{l}, \quad T_2 = \frac{1}{2T_0}\frac{k_2 - k_1}{M_2 + l}, \quad T_3 = \frac{1}{2T_0}\frac{k_2}{M_1 - l} \tag{3.2}$$

$$\Sigma T = T_1 + T_2 + T_3 \tag{3.3}$$

$$H = \frac{1}{\Sigma T}(T_1T_3p_1^2 + T_1T_2p_2^2 + T_2T_3p_3^2) \tag{3.4}$$

$$K_{ij} = M_i p_j - M_j p_i. \tag{3.5}$$

Note the constraint $lT_1 + (M_2 + l)T_2 - (M_1 - l)T_3 = 0$, which implies that for fixed l, only two of the T's are independent. Also, momentum conservation implies that K_{ij} is cyclically symmetric and we therefore use $K \equiv K_{12} = K_{23} = K_{31}$.

In general one must take the continuum limit with care because of ultraviolet and infrared divergences that might be present. Since we expect no ultraviolet divergences we can immediately take $a \to 0$. However, it turns out there are no infrared divergences either and we can see this by taking the continuum limit $a \to 0$ and $m \to 0$ simultaneously. In the $a \to 0$ limit we can replace the sums over k_1, k_2' for $k_1 > 0$ by integrals over T_1 and T_2 and in the $m \to 0$ limit we can replace the sum over l by an integral over T_3. The continuous transformation between T_1, T_2, T_3 and ak_1, ak_2, ml is given by Eq. (3.2) and the Jacobian is $\Sigma T/4m|l|(mM_1 - ml)(mM_2 + ml)$. Notice that integrating the T's from 0 to $+\infty$ accounts for summing over the *whole* range; $l > 0, k_1 > 0$ *and* $l < 0, k_1 < 0$. We obtain

$$\Gamma^{\wedge\wedge\wedge}=\left(\frac{g}{T_0}\right)\frac{g^2}{\pi^2}\frac{(K^\wedge)^3}{M_1M_2M_3}\times\int dT_1dT_2dT_3\frac{e^{-H(T_1,T_2,T_3)}T_1T_2T_3}{(T_1+T_2+T_3)^5} \tag{3.6}$$

$$=\left(\frac{g}{T_0}\right)\frac{g^2}{\pi^2}\frac{(K^\wedge)^3}{M_1M_2M_3}\int_0^\infty dx \tag{3.7}$$

$$\times\int_0^\infty dy\frac{xy}{(xp_1^2+xyp_2^2+yp_3^2)(1+x+y)^4} \tag{3.8}$$

where we have scaled $x=T_1/T_3$ and $y=T_2/T_3$ and note that $H(T_1,T_2,T_3)=T_3H(x,y,1)$. Notice that for all p_i's off-shell the double integral converges. For the special off-shell point $q^2\equiv p_1^2=p_2^2=p_3^2$ we get

$$\Gamma^{\wedge\wedge\wedge}=\left(\frac{g}{T_0}\right)\frac{g^2}{\pi^2}\frac{(K^\wedge)^3}{M_1M_2M_3}\frac{\chi}{q^2} \tag{3.9}$$

where

$$\chi\equiv\int_0^\infty dx\int_0^\infty dy\frac{xy}{(x+xy+y)(1+x+y)^4}\approx 0.030080945\ldots. \tag{3.10}$$

IV. TRIANGLE CONTRIBUTING TO CHARGE RENORMALIZATION

We turn now to the main task of this paper, the calculation of one-loop diagrams that contribute to charge renormalization. The self-energy diagrams have been evaluated in [9], so it remains to calculate the vertex corrections, i.e. the three transverse gluon triangle diagram. The kinematics of the three gluons are chosen as before. We start with the expressions for the diagrams that emerge after integrating over the transverse loop momentum. In this section we describe the analysis of the remaining sums over two discretized times, k_1a,k_2a and one discretized loop momentum $p^+=lm$ in the continuum limit, $a,m\rightarrow 0$.

A. All internal lines transverse

We denote the complete three transverse gluon vertex with polarization labels $\wedge,\wedge,\vee$ for gluons 1,2,3 by $\Gamma^{\wedge\wedge\vee}(p_1,p_2,p_3)$. The triangle diagrams displayed in Fig. 5, which have only transverse gluons on the internal lines, produce the following expression:

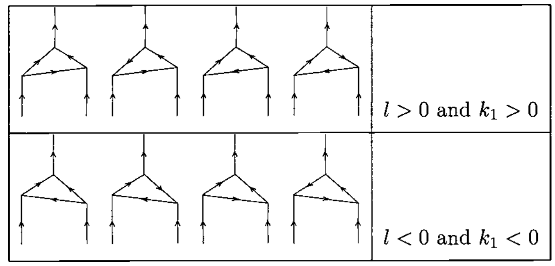

FIG. 5. Diagrams with all internal lines transverse.

$$\Gamma_I^{\wedge\wedge\vee}=\frac{g^3K^\wedge}{16\pi^2T_0^3}\frac{M}{M_1M_2}\sum_{l,k_1,k_2}\frac{e^{-H}}{|l|(M_2+l)(M_1-l)}\times\left(\frac{T_1T_2T_3K^2A}{M^2(T_1+T_2+T_3)^4}+\frac{T_1B_1+T_2B_2+T_3B_3}{(T_1+T_2+T_3)^3}\right) \tag{4.1}$$

where K,H and the T_i's are as before and

$$A=\frac{M^2M_1^2}{l^2(M_1-l)^2}+\frac{M^2M_2^2}{l^2(M_2+l)^2}+\frac{(M_2+l)^2}{(M_1-l)^2}+\frac{(M_1-l)^2}{(M_2+l)^2} \tag{4.2}$$

$$B_1=\frac{M_2M_1^3}{l^2(M_1-l)^2}+\frac{M_1M_2^3}{l^2(M_2+l)^2} \tag{4.3}$$

$$B_2=-\frac{MM_2^3}{l^2(M_2+l)^2}-\frac{M_2(M_2+l)^2}{M(M_1-l)^2}-\frac{M_2(M_1-l)^2}{M(M_2+l)^2} \tag{4.4}$$

$$B_3=-\frac{MM_1^3}{l^2(M_1-l)^2}-\frac{M_1(M_2+l)^2}{M(M_1-l)^2}-\frac{M_1(M_1-l)^2}{M(M_2+l)^2}. \tag{4.5}$$

The triangles with only two transverse gluon internal lines, denoted $\Gamma_{II}^{\wedge\wedge\vee}$ will be dealt with in the following subsection. We introduce the following notation that will help streamline some of the formulas: $P_i^*\equiv p_i^2/M_i$, $P^*\equiv p^2/M=-P_3^*$. For example,

$$K^2=-M_1M_2M_3(P_1^*+P_2^*+P_3^*)=M_1M_2M(P_1^*+P_2^*-P^*). \tag{4.6}$$

The vertex function should be antisymmetric under the interchange $p_1,M_1\leftrightarrow p_2,M_2$. In view of the explicit overall factor of $K^\wedge$ which is odd under this transformation, the coefficient of $K^\wedge$ should be even. Inspection of the above expression for the triangle diagram shows that this symmetry is realized in the following way. The expression for the summation range $k_1=-k_1'$, $l=-l'<0$ is precisely equal to that for the range $k_1,l>0$ *with the interchange of variables*

$p_1, M_1 \leftrightarrow p_2, M_2$. Thus, it is only necessary to explicitly calculate for one time ordering, say $k_1, l>0$. Then adding to this the result of the interchange gives the complete answer.

We are now dealing with potentially ultraviolet divergent diagrams. To reveal the ultraviolet structure we consider the continuum limit in the order $a\to 0$ followed by $m\to 0$. Recall that $a\neq 0$ serves as our ultraviolet cutoff. In the $a\to 0$ limit we can attempt to replace the sums over k_1, k_2' (k_1', k_2) for $k_1>0$ $(k_1<0)$ by integrals over T_1 and T_2 (T_3) just as in the preceding section. Since we wish to keep l fixed in this first step, for the case $k_1>0$ we express T_3 in terms of T_1 and T_2: $T_3=[lT_1+(M_2+l)T_2]/(M_1-l)$. For the case $k_1<0$, it is more convenient to express T_2 in terms of T_1 and T_3: $T_2=[l'T_1+(M_1+l')T_3]/(M_2-l')$. We find $\Sigma T=(MT_2+M_1T_1)/(M_1-l)=(MT_3+M_2T_1)/(M_2-l')$. For the A term, this procedure encounters no obstacle, and we obtain (displaying explicitly the contribution for $k_1, l>0$)

$$\Gamma_A^{\wedge\wedge\vee}\to\frac{g^3K^{\wedge}}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\sum_l\int dT_1dT_2\frac{(M_1-l)^2T_1T_2(lT_1+(M_2+l)T_2)K^2A}{M^2(M_1T_1+MT_2)^4}e^{-H(T_1,T_2)}+(1\leftrightarrow 2)\right\}$$
$$=\frac{g^3K^{\wedge}}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\sum_l\int dT\frac{(M_1-l)^2T(lT+(M_2+l))K^2A}{M^2H(T,1)(M_1T+M)^4}+(1\leftrightarrow 2)\right\}. \tag{4.7}$$

It will be useful to note that H can be written in the alternative forms

$$H=(M_2+l)T_2P^*+lT_1P_1^*+\frac{K^2}{M_1M}\left[\frac{(M_1-l)T_1T_2}{MT_2+M_1T_1}\right] \tag{4.8}$$

$$=(M_1+l')T_3P^*+l'T_1P_2^*+\frac{K^2}{M_2M}\left[\frac{(M_2-l')T_1T_3}{MT_3+M_2T_1}\right], \tag{4.9}$$

where the first is useful when $k_1, l>0$ and the second for $k_1, l<0$.

However, the B terms produce logarithmically divergent integrals with this procedure, so they must be handled differently. To deal with these logarithmically divergent terms, we first note the identities

$$\frac{T_1}{(T_1+T_2+T_3)^3}=-\frac{\partial}{\partial T_2}\frac{(M_1-l)^3}{2M}\frac{T_1}{(MT_2+M_1T_1)^2} \tag{4.10}$$

$$=-\frac{\partial}{\partial T_3}\frac{(M_2-l')^3}{2M}\frac{T_1}{(MT_3+M_2T_1)^2} \tag{4.11}$$

$$\frac{T_2}{(T_1+T_2+T_3)^3}=-\frac{\partial}{\partial T_2}\frac{(M_1-l)^3}{2M^2}\frac{M_1T_1+2MT_2}{(MT_2+M_1T_1)^2} \tag{4.12}$$

$$\frac{T_3}{(T_1+T_2+T_3)^3}=-\frac{\partial}{\partial T_3}\frac{(M_2-l')^3}{2M^2}\frac{M_2T_1+2MT_3}{(MT_3+M_2T_1)^2}, \tag{4.13}$$

where the partial derivatives are taken with T_1 fixed.

Because of the divergences we cannot immediately write the continuum limit of the B terms as an integral. However, we can make the substitution $e^{-H}\to(e^{-H}-e^{-H_0})+e^{-H_0}$, where H_0 is chosen to be an appropriate simplified version of H, which coincides with H at $T_2=0$. For $k_1, l>0$, it is convenient to choose $H_0=[lT_1+(M_2+l)T_2]P_1^*$, whereas for $k_1, l<0$ $H_0'=[l'T_1+(M_1+l')T_3]P_2^*$ is more convenient. Then the factor $(e^{-H}-e^{-H_0})$ regulates the integrand at small T_i so that the sums may then safely be replaced by integrals. We shall denote the contributions from these terms by $\Gamma_{B1}^{\wedge\wedge\vee}$. Then using the above identities, an integration by parts (for which the surface term vanishes) makes the integrand similar to that in $\Gamma_A^{\wedge\wedge\vee}$ and simplifications can be achieved (for details see Appendix A1):

$$\Gamma_A^{\wedge\wedge\vee}+\Gamma_{B1}^{\wedge\wedge\vee}\to\frac{g^3K^{\wedge}}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\sum_{l=1}^{M_1-1}\int_0^\infty dT\left[\frac{TK^2(M_1-l)^2A'}{M^2H(M+M_1T)^3}+\frac{lT(H_0-H)(M_1-l)M_1A'}{M^2H(lT+M_2+l)(M+M_1T)}\right.\right.$$
$$\left.\left.-\frac{lT(H_0-H)(M_1-l)M_1A}{M^2H(M+M_1T)^2}\right]+(1\leftrightarrow 2)\right\}, \tag{4.14}$$

where

$$A'=\frac{M^2M_2^2}{lM_1(M_2+l)^2}-\frac{M_2(M_2+l)^2}{M_1M(M_1-l)}-\frac{M_2(M_1-l)^3}{M_1M(M_2+l)^2}. \tag{4.15}$$

Since the integrand of Eq. (4.14) is a rational function of T the last integral can also be done. We sketch the evaluation in Appendix B.

There remains the contribution of the term e^{-H_0} which would give a divergent integral. However, because the T_1, T_2 (T_1, T_3) dependence in the exponential is disentangled by our choice of H_0, the sums can be directly analyzed in the $a\to 0$ limit, giving an explicit expression for the divergent part in terms of the lattice cutoff. We denote this contribution, containing the ultraviolet divergence of the triangles, by $\Gamma_{B2}^{\wedge\wedge\vee}$. Referring to Appendix A2 for details we obtain

$$\begin{aligned}\Gamma_{B2}^{\wedge\wedge\vee}=&\frac{g^3K^{\wedge}}{8\pi^2T_0}\frac{M}{M_1M_2}\Bigg(\Bigg\{\sum_{l=1}^{M_1-1}\frac{M_1-l}{MM_1}\left(\frac{N_1 l}{M_1}+\frac{N_2(M_2+l)}{M}\right)\left[\ln\frac{2p_1^+}{ap_1^2}+f\left(\frac{\alpha}{\beta}\right)\right]\\&-\sum_{l=1}^{M_1-1}\frac{M_1-l}{MM_1}\left(\frac{N_1 l}{M_1}-\frac{N_2(M_2+l)}{M}\right)\frac{\alpha}{\beta}f'\left(\frac{\alpha}{\beta}\right)\Bigg\}+(1\leftrightarrow 2)\Bigg)\\=&-\frac{g^3K^{\wedge}}{4\pi^2T_0}\frac{M}{M_1M_2}\left(\sum_{l=1}^{M_1-1}\left\{\frac{B'}{M}\left[\ln\frac{2p_1^+}{ap_1^2}+f\left(\frac{\alpha}{\beta}\right)\right]+\frac{AM_2(M_1-l)^2}{M^3M_1}\frac{\alpha}{\beta}f'\left(\frac{\alpha}{\beta}\right)\right\}+(1\leftrightarrow 2)\right),\end{aligned} \tag{4.16}$$

where we have defined

$$B'=\frac{(M_1-l)^3}{M^2(M_2+l)}+\frac{MM_1}{l(M_1-l)}+\frac{(M_2+l)^3}{M^2(M_1-l)} \tag{4.17}$$

and

$$N_1\equiv\frac{B_1(M_1-l)}{l}+B_3,\quad N_2\equiv\frac{B_2(M_1-l)}{M_2+l}+B_3 \tag{4.18}$$

$$f(x)=\frac{\ln x}{1-x}-x\int_0^\infty dt e^{-xt}\frac{1-xt-e^{-xt}}{(1-e^{-xt})^2}\ln(1-e^{-t}) \tag{4.19}$$

$$\alpha\equiv\frac{M_1}{l},\quad \beta\equiv\frac{M}{M_2+l}. \tag{4.20}$$

In $\Gamma_{B2}^{\wedge\wedge\vee}$ we can further simplify the term proportional to $\ln(2p_1^+/ap_1^2)$, which contains the ultraviolet divergence of the triangle diagrams. We obtain

$$\begin{aligned}\Gamma_{\rm div}^{\wedge\wedge\vee}=&-\frac{g^3K^{\wedge}}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\sum_{l=1}^{M_1-1}\frac{1}{M}\left[\frac{(M_1-l)^3}{M^2(M_2+l)}+\frac{MM_1}{l(M_1-l)}+\frac{(M_2+l)^3}{M^2(M_1-l)}\right]\ln\frac{2p_1^+}{ap_1^2}+(1\leftrightarrow 2)\right\}\\=&-\frac{g^3K^{\wedge}}{4\pi^2T_0}\frac{M}{M_1M_2}\Bigg\{\ln\frac{2p_1^+}{ap_1^2}\Bigg[\psi(M_1+M_2)-\psi(M_2+1)+3\psi(M_1)+3\gamma\\&+\frac{M_1-1}{M^3}\left(-\frac{11}{3}M_1^2-7M_1M_2-4M_2^2+\frac{M_1}{3}\right)\Bigg]+(1\leftrightarrow 2)\Bigg\}\end{aligned} \tag{4.21}$$

where ψ is the digamma function.

Writing out the terms from interchanging $1\leftrightarrow 2$ in this expression, and simplifying we obtain

$$\Gamma_{\text{uv}}^{\wedge\wedge\vee}=\frac{g^3K^{\wedge}}{16\pi^2T_0}\frac{M}{M_1M_2}\left\{\left[\ln\frac{2p_1^+}{ap_1^2}+\ln\frac{2p_2^+}{ap_2^2}\right]\left[-4[\psi(M)+\psi(M_1)+\psi(M_2)+3\gamma]+2\left(\frac{11}{3}-\frac{8}{M}+\frac{M}{M_1M_2}+\frac{2M_1M_2}{M^3}+\frac{1}{3M^2}\right)\right]\right.$$
$$\left.-\ln\frac{p_1^+p_2^2}{p_+p_1^2}\left[8[\psi(M_1)-\psi(M_2)]+\frac{2(M_1-M_2)}{3M^3}(-11M^2+2M_1M_2-1)-\frac{M_1-M_2}{M_1M_2}\right]\right\}$$
$$\to\frac{g^3K^{\wedge}}{16\pi^2T_0}\frac{M}{M_1M_2}\left\{2\sum_{i=1}^{3}\ln\frac{2|p_i^+|}{ap_i^2}\left[-4(\ln|M_i|+\gamma)+\frac{22}{9}\right]+\left[\ln\frac{p^2p_1^+}{p^+p_1^2}+\ln\frac{p^2p_2^+}{p^+p_2^2}\right]\left[-4(\ln M+\gamma)+\frac{22}{9}\right]\right.$$
$$\left.-4\left(\ln\frac{M_1}{M_2}\right)\ln\frac{p_1^2p_2^+}{p_1^+p_2^2}-\ln\frac{p_1^+p_2^2}{p_2^+p_1^2}\left[8\ln\frac{p_1^+}{p_2^+}+\frac{2(p_1^+-p_2^+)}{3p^+}\left(-11+\frac{2p_1^+p_2^+}{p^{+2}}\right)\right]\right\} \tag{4.22}$$

where the final expression, valid at large M,M_1,M_2, has been arranged so that the uv divergence appears symmetrically among the three legs of the vertex.

Putting everything together, the amplitude for the triangle with only transverse internal lines is given in the continuum limit by

$$\Gamma_I^{\wedge\wedge\vee}=\frac{g^3K^{\wedge}}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\frac{1}{M_1}\sum_{l=1}^{M_1-1}\left[\int_0^\infty dTI_1+S_1\right]+(1\leftrightarrow2)\right\}+\frac{g^3K^{\wedge}}{16\pi^2T_0}\frac{M}{M_1M_2}\left\{2\sum_{i=1}^{3}\ln\frac{2|p_i^+|}{ap_i^2}\left[-4(\ln|M_i|+\gamma)+\frac{22}{9}\right]\right.$$
$$\left.+\left[\ln\frac{p^2p_1^+}{p^+p_1^2}+\ln\frac{p^2p_2^+}{p^+p_2^2}\right]\left[-4(\ln M+\gamma)+\frac{22}{9}\right]-\ln\frac{p_1^+p_2^2}{p_2^+p_1^2}\left[4\ln\frac{p_1^+}{p_2^+}+\frac{2(p_1^+-p_2^+)}{3p^+}\left(-11+\frac{2p_1^+p_2^+}{p^{+2}}\right)\right]\right\} \tag{4.23}$$

where

$$I_1=I\left(T,p_1,p_2,\frac{l}{M_1}\right)$$
$$\equiv\frac{M_1(M_1-l)^2TK^2A'}{M^2H(T,1)(M_1T+M)^3}+\frac{lT(H_0-H)(M_1-l)M_1^2A'}{M^2H(lT+M_2+l)(M+M_1T)}-\frac{lT(H_0-H)(M_1-l)M_1^2A}{M^2H(M+M_1T)^2} \tag{4.24}$$

$$S_1=S\left(M_1,M_2,\frac{l}{M_1}\right)$$
$$\equiv-\left[\frac{M_1B'}{M}f\left(\frac{\alpha}{\beta}\right)+\frac{AM_2(M_1-l)^2}{M^3}\frac{\alpha}{\beta}f'\left(\frac{\alpha}{\beta}\right)\right] \tag{4.25}$$

$$I_2=I\left(T,p_2,p_1,\frac{l}{M_2}\right),\quad S_2=S\left(M_2,M_1,\frac{l}{M_2}\right) \tag{4.26}$$

and where we recall, for convenience, our definitions (appropriate to the case $k_1,l>0$)

$$A=\frac{M^2M_1^2}{l^2(M_1-l)^2}+\frac{M^2M_2^2}{l^2(M_2+l)^2}+\frac{(M_2+l)^2}{(M_1-l)^2}+\frac{(M_1-l)^2}{(M_2+l)^2}$$
$$A'=\frac{M^2M_2^2}{lM_1(M_2+l)^2}-\frac{M_2(M_2+l)^2}{M_1M(M_1-l)}-\frac{M_2(M_1-l)^3}{M_1M(M_2+l)^2}$$
$$B'=\frac{(M_2+l)^3}{M^2(M_1-l)}+\frac{(M_1-l)^3}{M^2(M_2+l)}+\frac{MM_1}{l(M_1-l)}$$
$$H=H(T,1)=(M_2+l)P_3^*+lTP_1^*+\frac{(M_1-l)T}{M+M_1T}\frac{K^2}{M_1M}$$
$$H_0=H_0(T,1)=(M_2+l)P_1^*+lTP_1^*$$
$$\frac{\alpha}{\beta}=\frac{M_1(M_2+l)}{lM}. \tag{4.27}$$

To complete the continuum limit we assume M,M_1,M_2 large and attempt to replace the sums over l by integrals over a continuous variable $\xi=l/M_1$, with $0<\xi<1$. This procedure is obstructed by the singular behavior of the integrand for ξ near 0 or 1. When this occurs, we introduce a cutoff $\epsilon\ll1$, and only do the replacement for $\epsilon<\xi<1-\epsilon$, dealing with the sums directly in the singular regions. The detailed analysis is presented in the Appendixes. Referring to Eq. (B30), we see that we can write

$$\Gamma_I^{\wedge\wedge\vee}=\Gamma_{I,\text{finite}}^{\wedge\wedge\vee}+\frac{g^3K^{\wedge}}{16\pi^2T_0}\frac{M}{M_1M_2}\Bigg\{2\sum_{i=1}^{3}\ln\frac{2|p_i^+|}{ap_i^2}\left[-4(\ln|M_i|+\gamma)+\frac{22}{9}\right]+\ln\frac{p^2p_1^+}{p^+p_1^2}\left[-4\ln\frac{M}{M_1}+\frac{22}{9}\right]+\ln\frac{p^2p_2^+}{p^+p_2^2}\left[-4\ln\frac{M}{M_2}+\frac{22}{9}\right]\Bigg\}$$
$$+\frac{g^3K^{\wedge}}{4\pi^2T_0}\Bigg(\frac{2\pi^2}{3}+\frac{M}{M_1M_2}\Bigg[(\ln M_1+\gamma)\Bigg\{\left(\frac{M_1p^2+Mp_1^2}{M_1p^2-Mp_1^2}+\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_1p^2}{Mp_1^2}-3+\frac{\pi^2}{6}\Bigg\}$$
$$+(\ln M_2+\gamma)\Bigg\{\left(\frac{M_2p^2+Mp_2^2}{M_2p^2-Mp_2^2}-\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_2p^2}{Mp_2^2}-3+\frac{\pi^2}{6}\Bigg\}\Bigg]\Bigg). \qquad (4.28)$$

B. Charge renormalization

As we have mentioned, $\Gamma_I^{\wedge\wedge\vee}$ contains the complete ultraviolet divergence in the one-loop vertex function, so we pause to discuss how the coupling renormalizes. Comparing the zeroth order vertex, $-2gK^{\wedge}M/M_1M_2T_0$, to Eq. (4.28), we see that the ultraviolet divergence of the triangle is contained in the multiplicative factor

$$1+\frac{g^2}{16\pi^2}\ln\frac{2}{a}\left[4(\ln M+\ln M_1+\ln M_2+3\gamma)-\frac{22}{3}\right]. \qquad (4.29)$$

Note the entanglement of ultraviolet $[\ln(1/a)]$ and infrared ($\ln M_i$) divergences, typical of light cone gauge. The $\ln M$'s multiplying $\ln(1/a)$ must cancel to give the correct charge renormalization. To see how this happens, recall that the self-energy calculation of [9] implies the gluon wave function renormalization factor

$$Z(Q)=1-\frac{g^2N_c}{16\pi^2}\left\{\left[8(\ln M+\gamma)-\frac{22}{3}\right]\ln\frac{2Q^+}{aQ^2}-\frac{4}{3}\right\}. \qquad (4.30)$$

Thus the appropriate wave function renormalization factor for the triangle, $\sqrt{Z(p_1)Z(p_2)Z(p)}$, contains the ultraviolet divergent factor

$$1-\frac{g^2N_c}{16\pi^2}[4(\ln MM_1M_2+3\gamma)-11]\ln\frac{2}{a}, \qquad (4.31)$$

so the divergence for the renormalized triangle is contained in the multiplicative factor

$$1+\frac{11}{3}\frac{g^2N_c}{16\pi^2}\ln\frac{2}{a} \qquad (4.32)$$

implying the correct relation of renormalized to bare charge

$$g_R=g\left(1+\frac{11}{24\pi}\alpha_sN_c\ln\frac{2}{a}\right), \qquad (4.33)$$

where $\alpha_s=g^2/2\pi$.

It is interesting to note that if the calculation is organized differently there is no entanglement of infrared and ultraviolet divergences. As suggested by the recent worldsheet approach to planar diagrams [20], it is natural to combine self-energy and vertex diagrams at each value of the discrete p^+ of the loop. For example, consider the diagrams of Fig. 6, which contribute to renormalization. It then turns out that the terms $1/l_0$, $1/(M-l_0)$, $1/(M_1-l_0)$, whose sums give rise to the $\ln M$'s, cancel among the three diagrams *before* the p^+ sum is done. With the arithmetic organized this way the entangled divergences never arise.

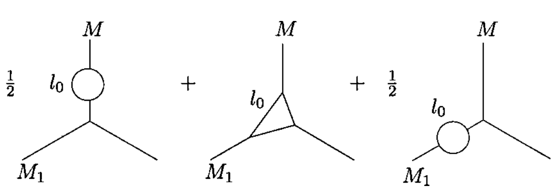

FIG. 6. Diagrams contributing to the renormalization. The l_0 next to a line denotes the discrete p^+ of that line. The three diagrams shown naturally go together for fixed l_0.

C. Longitudinal internal gluons

In addition to the diagrams with three transverse gluon internal lines discussed in the preceding section, there are diagrams where one of the internal lines is a longitudinal gluon (solid line with no arrow) or a fictitious gluon, whose exchange represents the four gluon vertex as the concatenation of two cubic vertices (dashed line). The various possibilities are shown in Fig. 7.

Again we start with the expression for the sum of these diagrams obtained after doing the transverse momentum integrals:

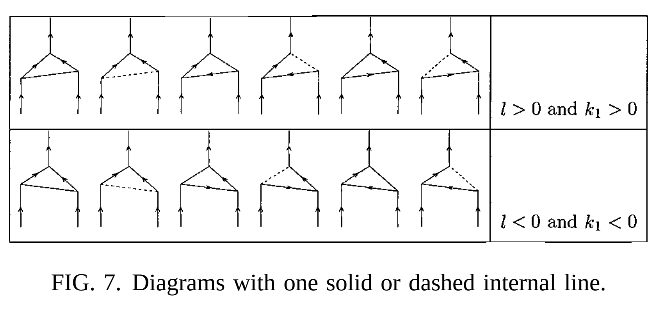

FIG. 7. Diagrams with one solid or dashed internal line.

$$\Gamma_{II}^{\wedge\wedge\vee} = -\frac{g^3}{16\pi^2 T_0^2} K^{\wedge} \sum_l \frac{1}{l(M_2+l)(M_1-l)} e^{-H} \times \left(\frac{T_1}{(T_1+T_2+T_3)^2} C_1 + \frac{T_2}{(T_1+T_2+T_3)^2} C_2 + \frac{T_3}{(T_1+T_2+T_3)^2} C_3 \right), \tag{4.34}$$

where

$$C_1 = f_{k_1} \frac{M_3(2M_1-l)(2M_2+l)}{l(M_1-l)(M_2+l)} + h_{k_1} \frac{M_3 l}{(M_2+l)(M_1-l)} \tag{4.35}$$

$$C_2 = f_{k_2-k_1} \frac{(M_2-l)(M_3+M_1-l)(M_1-l)}{|l|(M_2+l)M_1} + h_{k_2-k_1} \frac{(M_1-l)(M_2+l)}{M_1|l|} \tag{4.36}$$

$$C_3 = -f_{k_2} \frac{(M_1+l)(M_3+M_2+l)(M_2+l)}{|l|(M_1-l)M_2} - h_{k_2} \frac{(M_2+l)(M_1-l)}{|l|M_2}. \tag{4.37}$$

Since the f_k, h_k are assumed to fall off rapidly with k, the corresponding sum can never be replaced by an integral. Consider first the term with C_1. Then for $k_1>0$ (<0) only the k_2' (k_2) sum ranges freely from 1 to ∞. Writing out $H(T_1,T_2)$ explicitly in terms of k_1, k_2',

$$H = \frac{p^2 k_2'}{2MT_0} + \frac{p_1^2 k_1}{2M_1 T_0} + \frac{K^2 k_1}{2T_0 M M_1} \frac{(M_1-l)k_2'}{Mlk_2' + M_1 k_1 (M_2+l)}, \tag{4.38}$$

we see that, due to the limited range of k_1, the only term that can get large is the first one. Furthermore, all the other terms stay of order $O(a)$ in the limit $a\to 0$ at fixed m, which we are studying. Thus, writing $u = e^{-p^2/2MT_0}$, we see that we require the sum

$$\sum_{k_2'=1}^{\infty} \frac{u^{k_2'}}{(k_2'+z)^2} = \psi'(z+1) + O((1-u)\ln(1-u)) \tag{4.39}$$

in the $a\to 0$ limit. Thus the C_1 contribution has the $a\to 0$ limit

$$\Gamma_{C1}^{\wedge\wedge\vee} = -\frac{g^3}{8\pi^2 T_0} \frac{K^{\wedge}}{M} \left\{ \sum_{k=1}^{\infty} \sum_{l=1}^{M_1-1} \frac{k}{l} \psi'\left(1 + k\frac{M_1(M_2+l)}{lM}\right) \times \left[f_k \frac{(2M_1-l)(2M_2+l)}{l^2} + h_k \right] + (1\leftrightarrow 2) \right\}. \tag{4.40}$$

For the C_2 contribution, k_2' is limited by f,h. Then k_1 ranges freely from 1 to ∞ for $l>0$, but for $l<0$, $1 \leqslant k_1' \leqslant k_2'-1$. In the first case, only the second term of H can get large, and in the second case no term can get large. So with $u_1 = e^{-p_1^2/2M_1T_0}$, we need Eq. (4.39) (with $u\to u_1$ and $k_2' \to k_1'$) for $l>0$ and

$$\sum_{k_1'=1}^{k_2'-1} \frac{1}{(k_1'+z)^2} = \psi'(z+1) - \psi'(z+k_2'+1) - \frac{1}{(k_2'+z)^2} \quad \text{for } l<0. \tag{4.41}$$

The C_3 contribution is similar but with the roles of $l>0$ and $l<0$ switched. In fact, inspection shows that interchanging $1\leftrightarrow 2$ takes the C_2 contribution for $l>0$ $(l<0)$ into the C_3 contribution for $l<0$ $(l>0)$. So we need only display the $l>0$ cases explicitly. Combining all three contributions we have for $a\to 0$

$$\begin{aligned}\Gamma_{C1}^{\wedge\wedge\vee} + \Gamma_{C2}^{\wedge\wedge\vee} + \Gamma_{C3}^{\wedge\wedge\vee} \approx & -\frac{g^3}{8\pi^2 T_0} \frac{K^{\wedge}}{M} \left\{ \sum_{k=1}^{\infty} \sum_{l=1}^{M_1-1} \frac{k}{l} \psi'\left(1+k\frac{M_1(M_2+l)}{lM}\right) \left[f_k \frac{(2M_1-l)(2M_2+l)}{l^2} + h_k \right] \right. \\ & + \sum_{k=1}^{\infty} \sum_{l=1}^{M_1-1} \frac{k(M_1-l)^2 M}{(M_2+l)M_1^3} \psi'\left(1+k\frac{lM}{M_1(M_2+l)}\right) \left[f_k \frac{(M_2-l)(M+M_1-l)}{(M_2+l)^2} + h_k \right] \\ & + \sum_{k=1}^{\infty} \sum_{l=1}^{M_1-1} \frac{k(M_2+l)^2 M}{(M_1-l)M_2^3} \left[\frac{M_2^2(M_1-l)^2}{k^2 M_1^2 (M_2+l)^2} + \psi'\left(1+k\frac{M_1(M_2+l)}{M_2(M_1-l)}\right) - \psi'\left(1+k\frac{lM}{M_2(M_1-l)}\right) \right] \\ & \left. \times \left[f_k \frac{(M_1+l)(M+M_2+l)}{(M_1-l)^2} + h_k \right] + (1\leftrightarrow 2) \right\}. \end{aligned} \tag{4.42}$$

The l sum can be rewritten and evaluated approximately and the details can be found in Appendix A3. This yields the result of the continuum limit of the triangle with internal longitudinal gluons:

$$\Gamma_{II}^{\wedge\wedge\vee} \approx -\frac{g^3}{8\pi^2 T_0}\frac{K^\wedge}{M}\Bigg\{\sum_{k=1}^{\infty}\int_\epsilon^1 d\xi\,\frac{k}{\xi}\,\psi'\left(1+k\,\frac{M_2+\xi M_1}{\xi M}\right)\left[f_k\frac{(2-\xi)(2M_2+\xi M_1)}{\xi^2 M_1}+h_k\right]$$
$$+\sum_{k=1}^{\infty}\int_0^1 d\xi\frac{k(1-\xi)^2 M}{M_2+\xi M_1}\psi'\left(1+k\frac{\xi M}{M_2+\xi M_1}\right)\left[f_k\frac{(M_2-\xi M_1)(M+M_1(1-\xi))}{(M_2+\xi M_1)^2}+h_k\right]$$
$$+\sum_{k=1}^{\infty}\int_0^{1-\epsilon} d\xi\frac{k(M_2+\xi M_1)^2 M}{(1-\xi)M_2^3}\left[\frac{M_2^2(1-\xi)^2}{k^2(M_2+\xi M_1)^2}+\psi'\left(1+k\frac{M_2+\xi M_1}{M_2(1-\xi)}\right)-\psi'\left(1+k\frac{\xi M}{M_2(1-\xi)}\right)\right]$$
$$\times\left[f_k\frac{(1+\xi)(M+M_2+\xi M_1)}{M_1(1-\xi)^2}+h_k\right]-\frac{2M^2}{\epsilon M_1 M_2}+(1\leftrightarrow 2)\Bigg\}$$
$$-\frac{g^3}{4\pi^2 T_0}\frac{K^\wedge}{M}\left[+\frac{2M\pi^2}{3}-\frac{M^2}{M_1M_2}(\ln\epsilon^2 M_1M_2+2\gamma)\left(3-\frac{\pi^2}{6}\right)\right]. \tag{4.43}$$

As $\epsilon\to 0$ the rhs approaches a finite result. Then the divergence as $M\to\infty$ is contained entirely in the last line. Thus we can write in the continuum limit

$$\Gamma_{II}^{\wedge\wedge\vee}=\Gamma_{II,\text{finite}}^{\wedge\wedge\vee}-\frac{g^3K^\wedge}{4\pi^2T_0}\left[\frac{2\pi^2}{3}-\frac{M}{M_1M_2}(\ln M_1M_2+2\gamma)\left(3-\frac{\pi^2}{6}\right)\right]. \tag{4.44}$$

Notice that these divergent terms cancel some of the divergent terms in $\Gamma_I^{\wedge\wedge\vee}$ leading to the complete vertex

$$\Gamma^{\wedge\wedge\vee}=\Gamma_{\text{finite}}^{\wedge\wedge\vee}+\frac{g^3}{16\pi^2T_0}\frac{(p_1^+p_2^\wedge-p_2^+p_1^\wedge)p^+}{p_1^+p_2^+}\left\{\ln\frac{p^*}{p_1^*}\left[-4\ln\frac{p^+}{p_1^+}+\frac{22}{9}\right]+\ln\frac{p^*}{p_2^*}\left[-4\ln\frac{p^+}{p_2^+}+\frac{22}{9}\right]\right\}$$
$$+\frac{g^3}{4\pi^2T_0}\frac{(p_1^+p_2^\wedge-p_2^+p_1^\wedge)p^+}{p_1^+p_2^+}\left\{\frac{1}{2}\sum_{i=1}^{3}\ln\frac{2}{a|p_i^*|}\left[-4(\ln|M_i|+\gamma)+\frac{22}{9}\right]+(\ln M_1+\gamma)\left[\left(\frac{p^*+p_1^*}{p^*-p_1^*}-\frac{p_1^*-p_2^*}{p_1^*+p_2^*}\right)\ln\frac{p^*}{p_1^*}\right]\right.$$
$$\left.+(\ln M_2+\gamma)\left[\left(\frac{p^*+p_2^*}{p^*-p_2^*}+\frac{p_1^*-p_2^*}{p_1^*+p_2^*}\right)\ln\frac{p^*}{p_2^*}\right]\right\} \tag{4.45}$$

where for further brevity we have defined the continuum limit finite variables $p_i^*\equiv P_i^*/m=p_1^2/p_1^+$.

As already discussed, the uv divergences in the second line are canceled up to the standard asymptotic freedom result by the wave function renormalization factors. The remaining divergences in the last two lines are only linear in $\ln M_i$, and are unavoidable in the off-shell amplitude. Notice that there is a special off-shell point, $p_1^*=p_2^*=-p^*$, for which they disappear. The finite part depends in detail on the choice of f_k,h_k. The requirement of Lorentz invariance is expected to limit these parameters sharply, if not overdetermine them. If the latter holds, additional counterterms will be required to achieve Lorentz invariance.

V. CONCLUSIONS

In this article we have extended previous work [9] by further exploring the discretized $SU(N_c)$ gauge theory proposed there. Although the discretized theory is completely regulated, there is no guarantee that gauge invariance has been respected. Since we have chosen a non-covariant gauge, violations of gauge invariance would show up as violations of Lorentz invariance in the continuum limit. It is therefore important to check whether it reproduces known weak coupling results. We have shown here that to one loop order we obtain the correct renormalization of the coupling, Eq. (4.33). The remaining infrared divergences shown in the last two lines of Eq. (4.45) must be considered in the context of a physical quantity. There is no reason *a priori* to expect these divergences to disappear until one considers fully on-shell, color singlet external states. It is, however, reassuring that only single logarithmic divergences appear. As pointed out earlier these divergences do disappear at the special off-shell point $p_1^2/p_1^+=p_2^2/p_2^+=p_3^2/p_3^+\neq 0$.

The simplest on-shell scattering process is gluon-gluon scattering. Thus the complete resolution of the remaining infra-red divergence issues at one loop must await the analysis of one-loop four gluon amplitudes, the obvious next step in this investigation. As confidence is gained that the discretized theory is faithful to the gauge invariant continuum theory, the application of the formalism to calculate QCD fishnet diagrams or to formulate a worldsheet description of QCD in the spirit of [19] becomes more compelling.

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APPENDIX A: DETAILS OF CALCULATIONS

1. Evaluation of $\Gamma_{B1}^{\wedge\wedge\vee}$

In the calculation of $\Gamma_{B1}^{\wedge\wedge\vee}$ we start by integrating by parts. This transfers the derivative to the factor $(e^{-H}-e^{-H_0})$. For definiteness take the case $l>0$. Then we compute

$$\frac{\partial}{\partial T_2}(e^{-H}-e^{-H_0}) = -e^{-H}\frac{H-T_1lP_1^*}{T_2}+e^{-H_0}\frac{H_0-T_1lP_1^*}{T_2} + e^{-H}\frac{K^2}{M_1}\left[\frac{(M_1-l)T_1T_2}{(MT_2+M_1T_1)^2}\right]. \tag{A1}$$

The first two terms on the rhs partly cancel after integration over T_1,T_2. This is because the integrals are separately finite, so one can change variables $T_1=T_2T$ in each term separately. For the first term we find

$$-\int_0^\infty dT_1dT_2\mathcal{I}(T_1,T_2)\frac{H(T_1,T_2)-T_1lP_1^*}{T_2}e^{-H(T_1,T_2)} = -\int dTdT_2\mathcal{I}(T,1)[H(T,1)-TlP_1^*]e^{-T_2H(T,1)} = -\int dT\mathcal{I}(T,1)\left[1-\frac{TlP_1^*}{H(T,1)}\right], \tag{A2}$$

and the second term yields the same expression with $H(T,1)\to H_0(T,1)$, so the two terms combine to

$$\int_0^\infty dT\mathcal{I}(T,1)Tl\frac{H_0(T,1)-H(T,1)}{H(T,1)(lT+M_2+l)}. \tag{A3}$$

Simplifying the contribution to the T integrand from these terms leads to the continuum limit

$$\Gamma_{B1}^{\wedge\wedge\vee}\to\frac{g^3K^\wedge}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\sum_{l=1}^{M_1-1}\int_0^\infty dT\left[\frac{lT(H_0-H)}{H(lT+M_2+l)}+\frac{(M_1-l)TK^2}{HM_1(M+M_1T)^2}\right]\frac{(M_1-l)^2}{(M+M_1T)^2}\times\left[\frac{T}{2M}B_1+\frac{M_1T+2M}{2M^2}B_2+\frac{M_2T(M_1-l)+2MlT}{2M^2(M_1-l)}B_3+\frac{M_2+l}{M(M_1-l)}B_3\right]+(1\leftrightarrow2)\right\}$$
$$=\frac{g^3K^\wedge}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\sum_{l=1}^{M_1-1}\int_0^\infty dT\left[\frac{lT(H_0-H)}{H(lT+M_2+l)}+\frac{(M_1-l)TK^2}{HM_1(M+M_1T)^2}\right]\frac{(M_1-l)^2}{(M+M_1T)^2}\times\frac{M_1}{M^2}\left[\frac{M+M_1T}{M_1-l}A'-\frac{lT+M_2+l}{M_1-l}A\right]+(1\leftrightarrow2)\right\}, \tag{A4}$$

where we have defined

$$A'=\frac{M^2M_2^2}{lM_1(M_2+l)^2}-\frac{M_2(M_2+l)^2}{M_1M(M_1-l)}-\frac{M_2(M_1-l)^3}{M_1M(M_2+l)^2}. \tag{A5}$$

Notice that this result combines neatly with $\Gamma_A^{\wedge\wedge\vee}$ to give Eq. (4.14).

2. Evaluation of $\Gamma_{B2}^{\wedge\wedge\vee}$

We analyze the continuum limit of the $\Gamma_{B2}^{\wedge\wedge\vee}$ contribution to the B terms, which will be retained as discrete sums over the k's. Again for definiteness we display the case $l>0$ in detail:

$$\Gamma_{B2}^{\wedge\wedge\vee}=\frac{g^3K^\wedge}{16\pi^2T_0^3}\frac{M}{M_1M_2}\left\{\sum_{l=1}^{M_1-1}\frac{1}{|l|(M_2+l)(M_1-l)}\sum_{k_1,k_2'}\left[\frac{T_1B_1+T_2B_2+T_3B_3}{(T_1+T_2+T_3)^3}\right]e^{-H_0}+(1\leftrightarrow2)\right\}$$
$$=\frac{g^3K^\wedge}{4\pi^2T_0}\frac{M}{M_1M_2}\left\{\sum_{l=1}^{M_1-1}\frac{M_1-l}{|l|(M_2+l)}\sum_{k_1,k_2'}\left[\frac{k_1N_1+k_2'N_2}{(k_1\alpha+k_2'\beta)^3}\right]u_1^{k_1+k_2'}+(1\leftrightarrow2)\right\}, \tag{A6}$$

where

$$\alpha\equiv\frac{M_1}{l},\quad \beta\equiv\frac{M}{M_2+l},\quad N_1\equiv\frac{B_1(M_1-l)}{l}+B_3$$

$$N_2\equiv\frac{B_2(M_1-l)}{M_2+l}+B_3,\quad u_1\equiv e^{-p_1^2/2M_1T_0}=e^{-ap_1^2/2p_1^+},$$

$$u_2\equiv e^{-p_2^2/2M_2T_0}=e^{-ap_2^2/2p_2^+},\tag{A7}$$

where u_2 is to be used in the case $k_1,l<0$ instead of u_1. Clearly the continuum limit entails $u_1,u_2\to1$, causing the k sums to diverge logarithmically. To make this explicit, we first note the integral representation

$$\sum_{k_1,k_2'}\frac{u_1^{k_1+k_2'}}{(k_1\alpha+k_2'\beta)^2}=\int_0^\infty tdt\frac{u_1^2}{(e^{\alpha t}-u_1)(e^{\beta t}-u_1)}\tag{A8}$$

$$\approx\int_\epsilon^\infty tdt\frac{1}{(e^{\alpha t}-1)(e^{\beta t}-1)}+\int_0^\epsilon\frac{tdt}{(1-u_1+\alpha t)(1-u_1+\beta t)}\tag{A9}$$

where the approximate form is valid for $1-u_1\ll\epsilon\ll\alpha,\beta$. Doing the integral in the second term leads to

$$\sum_{k_1,k_2'}\frac{u_1^{k_1+k_2'}}{(k_1\alpha+k_2'\beta)^2}\approx\frac{1}{\alpha\beta}\left[\ln\frac{2p_1^+}{ap_1^2}+\frac{\beta\ln\alpha-\alpha\ln\beta}{\beta-\alpha}\right]+\frac{1}{\alpha\beta}\ln\epsilon+\int_\epsilon^\infty tdt\frac{1}{(e^{\alpha t}-1)(e^{\beta t}-1)}.\tag{A10}$$

The sums we require can be obtained from this identity by differentiation with respect to α or β. To present the results it is convenient to define a function $f(x)$ by

$$f(x)\equiv\frac{1}{2}\,\frac{1+x}{1-x}\ln x+\lim_{\epsilon\to0}\left[\ln\epsilon+\int_\epsilon^\infty\frac{tdt}{(e^{t\sqrt{x}}-1)(e^{t/\sqrt{x}}-1)}\right]\tag{A11}$$

$$=\frac{\ln x}{1-x}-x\int_0^\infty dte^{-xt}\frac{1-xt-e^{-xt}}{(1-e^{-xt})^2}\ln(1-e^{-t})\tag{A12}$$

where the second form is obtained by integration by parts. It is evident from the first form that $f(x)=f(1/x)$. Also one can easily calculate $f(1)=-\pi^2/6$. From the second form one easily sees that $f(x)\sim\ln x$ for $x\to0$, whence from the symmetry, $f(x)\sim-\ln x$ for $x\to\infty$. Exploiting the function f and its symmetries, we deduce

$$\sum_{k_1,k_2'}\frac{u_1^{k_1+k_2'}}{(k_1\alpha+k_2'\beta)^2}\approx\frac{1}{\alpha\beta}\left[\ln\frac{2p_1^+}{ap_1^2}+f\left(\frac{\alpha}{\beta}\right)\right]\tag{A13}$$

$$\sum_{k_1,k_2'}\frac{k_1u_1^{k_1+k_2'}}{(k_1\alpha+k_2'\beta)^2}\approx\frac{1}{2\alpha^2\beta}\left[\ln\frac{2p_1^+}{ap_1^2}+f\left(\frac{\alpha}{\beta}\right)\right]-\frac{1}{2\alpha\beta^2}\,f'\left(\frac{\alpha}{\beta}\right)\tag{A14}$$

$$\sum_{k_1,k_2'}\frac{k_2'u_1^{k_1+k_2'}}{(k_1\alpha+k_2'\beta)^2}\approx\frac{1}{2\alpha\beta^2}\left[\ln\frac{2p_1^+}{ap_1^2}+f\left(\frac{\alpha}{\beta}\right)\right]+\frac{1}{\beta^3}\,f'\left(\frac{\alpha}{\beta}\right)$$

$$\approx\frac{1}{2\alpha\beta^2}\left[\ln\frac{2p_1^+}{ap_1^2}+f\left(\frac{\beta}{\alpha}\right)\right]-\frac{1}{\alpha^2\beta}\,f'\left(\frac{\beta}{\alpha}\right).\tag{A15}$$

Inserting these results into Eq. (A6) produces Eq. (4.16).

3. Evaluation of $\Gamma_{II}^{\wedge\wedge\vee}$

For large M_i, the sum over l can be approximated by an integral over $\xi=l/M_1$ from $\epsilon\ll1$ to $1-\epsilon$, plus sums for $1\leqslant l\leqslant\epsilon M_1$ and $M_1(1-\epsilon)\leqslant l\leqslant M_1-1$ which contain the divergences. These divergences are only present in the first sum on the rhs of Eq. (4.42) for $l\ll M_1$ and in the last sum for $M_1-l\ll M_1$. The middle sum contains no divergence and can be replaced by an integral from 0 to 1 with no ϵ cutoff. To extract these divergent contributions, we can use the large argument expansion of ψ'

$$\psi'(z+1)\sim\frac{1}{z}-\frac{1}{2z^2}+O\left(\frac{1}{z^3}\right)\tag{A16}$$

to isolate them. It is thus evident that their coefficients will be proportional to the moments $\Sigma f_k/k^n$, $\Sigma h_k/k^n$ for $k=0,1$, which are precisely the moments constrained by the requirement that the gluon remain massless at one loop.

For the end point near $l=0$, we put $z=kM_1(M_2+l)/lM$ and write

$$\frac{k}{l}\psi'(z+1)\sim\frac{M}{M_1(M_2+l)}-\frac{l}{2k}\,\frac{M^2}{M_1^2(M_2+l)^2}+\ \dots,\tag{A17}$$

so the summand for small l becomes

$$\sum_k \left\{ f_k \frac{M(2M_1-l)(2M_2+l)}{l^2 M_1(M_2+l)} + \frac{h_k M}{M_1(M_2+l)} - \frac{f_k}{2k} \frac{M^2(2M_1-l)(2M_2+l)}{lM_1^2(M_2+l)^2} - \frac{h_k}{2k} \frac{lM^2}{M_1^2(M_2+l)^2} \right\}$$
$$\sim \frac{M(2M_1-l)(2M_2+l)}{l^2 M_1(M_2+l)} + \frac{M}{M_1(M_2+l)} - \frac{\pi^2}{12} \frac{M^2(2M_1-l)(2M_2+l)}{lM_1^2(M_2+l)^2} + \frac{\pi^2(l-1)}{36l} \frac{lM^2}{M_1^2(M_2+l)^2}$$
$$\sim \frac{4M}{l^2} - \frac{2M^2}{lM_1M_2} - \frac{\pi^2}{3} \frac{M^2}{lM_1M_2}. \tag{A18}$$

Summing l up to ϵM_1 gives

$$\sum_{l=1}^{\epsilon M_1} \left[\frac{4M}{l^2} - \frac{2M^2}{lM_1M_2} - \frac{\pi^2}{3} \frac{M^2}{lM_1M_2} \right] \approx \frac{4M\pi^2}{6} - \frac{4M}{\epsilon M_1} - \frac{M^2}{M_1M_2}(\ln \epsilon M_1 + \gamma)\left(2 + \frac{\pi^2}{3}\right). \tag{A19}$$

Inserting these results into Eq. (4.40) and writing out explicitly the $1 \leftrightarrow 2$ terms for the divergent part gives

$$\Gamma_{C1}^{\wedge\wedge\vee} \approx -\frac{g^3}{8\pi^2 T_0} \frac{K^\wedge}{M} \left\{ \sum_{k=1}^{\infty} \int_\epsilon^1 d\xi \frac{k}{\xi} \psi' \left(1 + k\frac{(M_2+\xi M_1)}{\xi M} \right) \left[f_k \frac{(2-\xi)(2M_2+\xi M_1)}{\xi^2 M_1} + h_k \right] + (1 \leftrightarrow 2) - \frac{4M^2}{\epsilon M_1 M_2} + \frac{8M\pi^2}{6} \right.$$
$$\left. - \frac{M^2}{M_1M_2}(\ln \epsilon^2 M_1 M_2 + 2\gamma)\left(2 + \frac{\pi^2}{3}\right) \right\}. \tag{A20}$$

Only the third sum contributes near $l = M_1$. We again use the large argument expansion of ψ'. But this time one only gets a logarithmic divergence, because the difference of ψ''s is of order $(M-l)^2$ as is the explicit rational term. Putting $z_1 = kM_1(M_2+l)/M_2(M_1-l)$ and $z_2 = klM/M_2(M_1-l)$, we have

$$\psi'(z_1+1) - \psi'(z_2+1)$$
$$\sim \left(\frac{1}{z_1} - \frac{1}{z_2} \right)\left(1 + \frac{1}{2z_1} + \frac{1}{2z_2} \right)$$
$$\sim -\frac{M_2^2(M_1-l)^2}{klMM_1(M_2+l)}[1 + O(M_1-l)]$$
$$\sim -\frac{M_2^2(M_1-l)^2}{kM_1^2 M^2}. \tag{A21}$$

Thus the $l \sim M_1$ end point divergence is just

$$-\frac{g^3}{8\pi^2} \frac{K^\wedge}{M} \frac{4M^2}{M_1M_2} \sum_{k=1}^{\infty} \left(\frac{f_k}{k} - f_k \right) \sum_{l=M_1(1-\epsilon)}^{M_1-1} \frac{1}{M_1-l}$$
$$= -\frac{g^3}{8\pi^2} \frac{4MK^\wedge}{M_1M_2} \left(\frac{\pi^2}{6} - 1 \right)(\ln \epsilon M_1 + \gamma). \tag{A22}$$

Putting everything together we obtain Eq. (4.43) for the continuum limit of the triangle with internal longitudinal gluons.

APPENDIX B: DIVERGENT PARTS OF INTEGRALS AND SUMS

The T integral in Eq. (4.14) can be evaluated by expanding the integrand

$$I \equiv \frac{M_1TK^2(M_1-l)^2A'}{M^2H(M+M_1T)^3} + \frac{lT(H_0-H)(M_1-l)M_1^2A'}{M^2H(lT+M_2+l)(M+M_1T)}$$
$$- \frac{lT(H_0-H)(M_1-l)M_1^2A}{M^2H(M+M_1T)^2} \tag{B1}$$

in partial fractions. First note that since $(M_1T+M)H$ is a quadratic polynomial, it may be factored as

$$(M_1T+M)H = lp_1^2(T-T_+)(T-T_-) \tag{B2}$$

where $T_- \sim -(K^2+M_1M_2p^2)/lMp_1^2$ and $T_+ \sim -MM_2p^2/(K^2+M_1M_2p^2)$ when $l \ll M_1$. Then the partial fraction expansion reads

$$I = \frac{R_1}{T-T_+} + \frac{R_2}{T-T_-} + \frac{R_3}{lT+M_2+l} + \frac{R_4}{(M_1T+M)^2}$$
$$+ \frac{R_5}{M_1T+M}, \tag{B3}$$

with the R_i independent of T. Of course the R_i are such that I falls off at least as $1/T^2$ for large T, i.e. $R_1+R_2+R_3/l+R_5/M_1=0$. This identity is helpful for determining R_5. Thus we have

$$\int_0^\infty dTI = -R_1\ln(-T_+) - R_2\ln(-T_-) + \frac{R_3}{l}\ln\frac{l}{M_2+l}$$
$$+ \frac{R_4}{MM_1} + \frac{R_5}{M_1}\ln\frac{M_1}{M}. \tag{B4}$$

The R_i are given explicitly by

$$R_1 = \frac{M_1(M_1-l)T_+}{lM^2p_1^2(T_+-T_-)}\left[\frac{(M_1-l)K^2A'}{(M_1T_++M)^2}+lp_1^2A' - \frac{lp_1^2(M_2+l+lT_+)A}{M_1T_++M}\right] \tag{B5}$$

$$R_2 = -\frac{M_1(M_1-l)T_-}{lM^2p_1^2(T_+-T_-)}\left[\frac{(M_1-l)K^2A'}{(M_1T_-+M)^2}+lp_1^2A' - \frac{lp_1^2(M_2+l+lT_-)A}{M_1T_-+M}\right] \tag{B6}$$

$$R_3 = -l\frac{(M_2+l)M_1^2A'}{M^2M_2} \tag{B7}$$

$$R_4 = M_1^2(M_1-l)\frac{A'}{M} - lM_1(M_1-l)\frac{A}{M} \tag{B8}$$

$$R_5 = \frac{lM_1(M_1-l)A}{M^2}\left(1-\frac{p_1^2MM_2}{K^2}\right)+\frac{lM_1^2A'}{MM_2} - \frac{M_1^2(M_1-l)A'}{M^2}\left[1-\frac{M}{M_1T_++M} - \frac{M}{M_1T_-+M}\right]. \tag{B9}$$

When the M's are large, the sum over l can be replaced by an integral over $\xi = l/M_1$ as long as ξ is kept away from the end points $\xi = 0,1$. We can isolate the terms that give rise to singular end point contributions and simplify them considerably. We shall then separate the divergent contributions and display them in detail.

First note that the worst end point divergence is $\xi^{-1}\ln\xi$ near $\xi=0$ or $1/(1-\xi)$ near $\xi=1$. Thus we can drop all terms down by a factor of l/M_i for small l or by $(M_1-l)/M_i$ for l near M_1. Thus for $l \ll M_i$, we note that $lT_-(M+M_1T_+)\sim -K^2/p_1^2$ and $T_+/(M+M_1T_+)\sim -M_2p^2/K^2$ and obtain

$$R_1 \sim -\frac{p^2M_1^2M_2}{lK^2}\frac{K^2+M_1M_2p^2-2M_2Mp_1^2}{(K^2+M_1M_2p^2)} = -\frac{p^2M_1^2M_2}{lK^2}\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2} \tag{B10}$$

$$R_2 \sim -\frac{M_1}{l}+\frac{2M_1M_2p_1^2}{l(M_1p_2^2+M_2p_1^2)} = -\frac{M_1}{l}\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2} \tag{B11}$$

$$R_3 \sim -M_1 \tag{B12}$$

$$R_4 \sim -\frac{MM_1^2}{l} \tag{B13}$$

$$R_5 \sim \frac{M_1^2}{l}\left[1+\frac{MM_1p_2^2-MM_2p_1^2}{K^2}\right]. \tag{B14}$$

Combining the $l\approx 0$ end point contributions gives

$$\int_0^\infty dTI \sim -R_1\ln\frac{MM_2p^2}{K^2+M_1M_2p^2} - R_2\ln\frac{K^2+M_1M_2p^2}{lMp_1^2}+\frac{R_3}{l}\ln\frac{l}{M_2}+\frac{R_4}{MM_1}+\frac{R_5}{M_1}\ln\frac{M_1}{M}$$

$$\sim \frac{M_1}{l}\left\{\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\left[\frac{M_1M_2p^2}{K^2}\ln\frac{MM_2p^2}{K^2+M_1M_2p^2}+\ln\frac{K^2+M_1M_2p^2}{lMp_1^2}\right] - \ln\frac{l}{M_2}+\left[1+\frac{MM_1p_2^2-MM_2p_1^2}{K^2}\right]\ln\frac{M_1}{M}-1\right\}$$

$$\sim \frac{M_1}{l}\left\{M\frac{M_1p_2^2-M_2p_1^2}{K^2}\ln\frac{M_1M_2p^2}{K^2+M_1M_2p^2}+\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\ln\frac{(K^2+M_1M_2p^2)^2}{lM^2M_2p^2p_1^2} - \ln\frac{lM}{M_1M_2}-1\right\} \quad \text{for } l \ll M_i. \tag{B15}$$

For the other end point, $M_1-l\ll M_i$, the roots of the polynomial $(M+M_1T)H$ approach $T_1=-M/M_1$ and $T_2=-p^2/p_1^2$. Which of these roots is approached by $T_\pm$ depends on the parameter values, but since the formulas are symmetric under their interchange, we can choose to use the first in place of T_+ and the second in place of T_-. Since the denominator $M+M_1T_1=0$ in this limit, we need to carefully evaluate

$$\frac{M_1-l}{M+M_1T_1}\sim-\frac{M_1(Mp_1^2-M_1p^2)}{K^2}.$$

Then we obtain for small M_1-l,

$$R_1\sim\frac{M_1M_2(Mp_1^2+M_1p^2)}{(M_1-l)K^2}-\frac{2MM_1p_1^2}{(M_1-l)(Mp_1^2-M_1p^2)} \tag{B16}$$

$$R_2\sim-\frac{2p^2M_1^2}{(M-l)(M_1p^2-Mp_1^2)} \tag{B17}$$

$$R_3\sim l\frac{M_1}{M_1-l} \tag{B18}$$

$$R_4\sim-\frac{2MM_1^2}{M_1-l} \tag{B19}$$

$$R_5\sim\frac{M_1^2}{(M_1-l)}-\frac{M_1^2M_2(Mp_1^2+M_1p^2)}{(M_1-l)K^2}. \tag{B20}$$

Combining the $l\approx M_1$ end point contributions gives

$$\int_0^\infty dTI\sim\left[R_1+\frac{R_3+R_5}{M_1}\right]\ln\frac{M_1}{M}-R_2\ln\frac{p^2}{p_1^2}+\frac{R_4}{MM_1}\quad\text{for}\ \ M_1-l\ll M_i$$
$$\sim\frac{2M_1}{(M_1-l)}\left[\frac{M_1p^2}{M_1p^2-Mp_1^2}\ln\frac{M_1p^2}{Mp_1^2}-1\right]. \tag{B21}$$

In writing the l sum as an integral these end point divergences can be separated by picking $\epsilon\ll1$ and summing l in the ranges $1\leqslant l\leqslant\epsilon M_1$ and $M_1(1-\epsilon)\leqslant l\leqslant M_1-1$. For these parts of the sum the above approximations can be made and the sum evaluated:

$$\sum_{l=1}^{\epsilon M_1}\frac{1}{l}=\sum_{l=M_1(1-\epsilon)}^{M_1-1}\frac{1}{M_1-l}=\psi(1+\epsilon M_1)+\gamma\sim\ln\epsilon M_1+\gamma$$

$$\sum_{l=1}^{\epsilon M_1}\frac{\ln l}{l}=-\gamma[\psi(1+\epsilon M)+\gamma]-\int_0^\infty dt\ln t\frac{e^{-t}-e^{-M_1\epsilon t}}{1-e^{-t}}$$
$$\sim\frac{1}{2}\ln^2(M_1\epsilon)+\frac{\zeta(2)-\gamma^2}{2}+\frac{1}{2}\int_0^\infty dt\frac{t\ln^2t}{e^t-1}$$
$$\equiv\frac{1}{2}\ln^2(M_1\epsilon)+C. \tag{B22}$$

The rest of the sum is replaced by an integral over $\epsilon\leqslant\xi\leqslant1-\epsilon$

$$\frac{1}{M_1}\sum_{l=1}^{M_1-1}\int_0^\infty IdT\sim\int_\epsilon^{1-\epsilon}d\xi\int_0^\infty IdT+(\ln\epsilon M_1+\gamma)\left[\left(\frac{2M_1p^2}{M_1p^2-Mp_1^2}+\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_1p^2}{Mp_1^2}-3+\left(M\frac{M_1p_2^2-M_2p_1^2}{K^2}-2\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_1M_2p^2}{K^2+M_1M_2p^2}\right]-\left[\ln\frac{M}{M_1M_2}(\ln\epsilon M_1+\gamma)+\frac{1}{2}\ln^2(\epsilon M_1)+C\right]\frac{2M_1p_2^2}{M_1p_2^2+M_2p_1^2}. \tag{B23}$$

Finally, we must extract the divergent contributions that arise from replacing the sums

$$\sum_{l=1}^{M_1-1}S_l=-\sum_{l=1}^{M_1-1}\left[\frac{M_1B'}{M}f\left(\frac{\alpha}{\beta}\right)+\frac{AM_2(M_1-l)^2}{M^3}\frac{\alpha}{\beta}f'\left(\frac{\alpha}{\beta}\right)\right] \tag{B24}$$

in Eq. (4.23) by an integral. First, for $l\approx M_1$, $\alpha/\beta\approx1$ and only the first term gives a singular end point contribution,

$$\sum_{l=M_1(1-\epsilon)}^{M_1-1}S_l\sim-2M_1f(1)[\psi(1+\epsilon M_1)+\gamma]\sim\frac{\pi^2}{3}M_1[\ln\epsilon M_1+\gamma]. \tag{B25}$$

On the other hand, for $l\approx0$, we have

$$f\left(\frac{M_1M_2}{lM}\right)\sim-\ln\frac{M_1M_2}{lM}-\int_0^\infty dt\,e^{-t}\ln t\frac{1-t-e^{-t}}{(1-e^{-t})^2}=\ln\frac{lM}{M_1M_2} \tag{B26}$$

$$\frac{M_1M_2}{lM}f'\left(\frac{M_1M_2}{lM}\right)\sim -1+\frac{lM}{M_1M_2}\left[\frac{\pi^2}{12}+\ln\frac{M_1M_2}{lM}-1\right]. \tag{B27}$$

The integral in the first line is zero because the integrand is a derivative of a function vanishing at the end points. Inserting these approximations, we obtain

$$\sum_{l=1}^{\epsilon M_1} S_l \sim \frac{M_1^2M_2}{M}\sum_{l=1}^{\epsilon M_1}\frac{2}{l^2}+M_1\left\{\sum_{l=1}^{\epsilon M_1}\frac{1}{l}\ln\frac{lM}{M_1M_2}-[\psi(1+\epsilon M_1)+\gamma]\frac{\pi^2}{6}\right\}. \tag{B28}$$

Putting Eqs. (B23), (B25), (B28) together, some simplification occurs and we obtain

$$\begin{aligned}\frac{1}{M_1}\sum_{l=1}^{M_1-1}\left[S(l/M_1)+\int_0^\infty IdT\right]\sim &\int_\epsilon^{1-\epsilon}d\xi\int_0^\infty IdT+\int_\epsilon^{1-\epsilon}d\xi S(\xi)+\frac{M_1M_2\pi^2}{3M}-\frac{2M_2}{\epsilon M}+(\ln\epsilon M_1+\gamma)\left[\left(\frac{2M_1p^2}{M_1p^2-Mp_1^2}\right.\right.\\ &\left.+\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_1p^2}{Mp_1^2}+\left(M\frac{M_1p_2^2-M_2p_1^2}{K^2}-2\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_1M_2p^2}{K^2+M_1M_2p^2}-3+\left.\frac{\pi^2}{6}\right]\\ &-\left[\ln\frac{M}{M_1M_2}(\ln\epsilon M_1+\gamma)+\frac{1}{2}\ln^2(\epsilon M_1)+C\right]\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}.\end{aligned} \tag{B29}$$

When we add the contribution with $1\leftrightarrow 2$, the antisymmetry of some of the coefficients leads to further simplification as well as a reduction in the degree of divergence of some of the terms:

$$\begin{aligned}&\frac{1}{M_1}\sum_{l=1}^{M_1-1}\left[S(l/M_1)+\int_0^\infty IdT\right]+(1\leftrightarrow 2)\\ &\sim\int_\epsilon^{1-\epsilon}d\xi\int_0^\infty IdT+\int_\epsilon^{1-\epsilon}d\xi S(\xi)+(1\leftrightarrow 2)-\frac{2}{\epsilon}-\ln\epsilon\frac{p_1^+p_2^2-p_2^+p_1^2}{p_1^+p_2^2+p_2^+p_1^2}\ln\frac{p_1^+}{p_2^+}\\ &\quad+\ln\epsilon\left[\frac{2p_1^+p^2}{p_1^+p^2-p^+p_1^2}\ln\frac{p_1^+p^2}{p^+p_1^2}+\frac{p_1^+p_2^2-p_2^+p_1^2}{p_1^+p_2^2+p_2^+p_1^2}\ln\frac{p_1^+p_2^2}{p_2^+p_1^2}+\frac{2p_2^+p^2}{p_2^+p^2-p^+p_2^2}\ln\frac{p_2^+p^2}{p^+p_2^2}-6+\frac{\pi^2}{3}\right]\\ &\quad+\ln\frac{p_1^+}{p_2^+}\left[\left(M\frac{M_1p_2^2-M_2p_1^2}{K^2}-2\frac{p_1^+p_2^2-p_2^+p_1^2}{p_1^+p_2^2+p_2^+p_1^2}\right)\ln\frac{M_1M_2p^2}{K^2+M_1M_2p^2}+\frac{1}{2}\ln\frac{p_1^+p_2^+}{M^2}\,\frac{p_1^+p_2^2-p_2^+p_1^2}{p_1^+p_2^2+p_2^+p_1^2}\right]\\ &\quad+\frac{2\pi^2M_1M_2}{3M}+\left[(\ln M_1+\gamma)\left\{\left(\frac{2M_1p^2}{M_1p^2-Mp_1^2}+\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_1p^2}{Mp_1^2}-3+\frac{\pi^2}{6}\right\}\right.\\ &\quad\left.+(\ln M_2+\gamma)\left\{\left(\frac{2M_2p^2}{M_2p^2-Mp_2^2}-\frac{M_1p_2^2-M_2p_1^2}{M_1p_2^2+M_2p_1^2}\right)\ln\frac{M_2p^2}{Mp_2^2}-3+\frac{\pi^2}{6}\right\}\right].\end{aligned} \tag{B30}$$

As $\epsilon\rightarrow 0$ the first two lines on the rhs approach a finite ϵ independent answer. The third line is explicitly finite. All divergences are shown in the last two lines. As $M_i\rightarrow\infty$ there is a leading linear divergence as well as a single logarithmic sub-leading divergence.

[1] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[2] E. Brezin, C. Itzykson, G. Parisi, and J.B. Zuber, Commun. Math. Phys. **59**, 35 (1978); Yu.M. Makeenko and A.A. Migdal, Phys. Lett. **88B**, 135 (1979); K. Bardakci, Nucl. Phys. **B193**, 145 (1981); T. Eguchi and H. Kawai, Phys. Rev. Lett. **48**, 1063 (1982); J. Greensite and M.B. Halpern, Nucl. Phys. **B211**, 343 (1983); A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D **27**, 2397 (1983).
[3] K.G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[4] M. Mueller-Preussker *et al.*, Contents of Lattice 2001 Proceedings, 2002, hep-lat/0203004.
[5] Proceedings of the 17th International Symposium on Lattice Field Theory, Pisa, Italy, 1999, edited by M. Campostrini *et al.*
[6] B. Lucini and M. Teper, J. High Energy Phys. **06**, 050 (2001), and references therein.
[7] H.B. Nielsen and P. Olesen, Phys. Lett. **32B**, 203 (1970); B. Sakita and M.A. Virasoro, Phys. Rev. Lett. **24**, 1146 (1970).
[8] J.M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[9] K. Bering, J.S. Rozowsky, and C.B. Thorn, Phys. Rev. D **61**, 045007 (2000).

[10] C.B. Thorn, Phys. Lett. **70B**, 85 (1977); Phys. Rev. D **17**, 1073 (1978).
[11] R. Giles, L. McLerran, and C.B. Thorn, Phys. Rev. D **17**, 2058 (1978).
[12] R. Brower, R. Giles, and C. Thorn, Phys. Rev. D **18**, 484 (1978).
[13] C.B. Thorn, Phys. Rev. D **20**, 1934 (1979).
[14] G. Curci, W. Furmanski, and R. Petronzio, Nucl. Phys. **B175**, 27 (1980).
[15] R.J. Perry, Phys. Lett. B **300**, 8 (1993).
[16] S. Mandelstam, Nucl. Phys. **B213**, 149 (1983).
[17] G. Leibbrandt, Phys. Rev. D **29**, 1699 (1984).
[18] A. Bassetto, M. Dalbosco, I. Lazzizzera, and R. Soldati, Phys. Rev. D **31**, 2012 (1985).
[19] K. Bardakci and C.B. Thorn, Nucl. Phys. **B626**, 287 (2002).
[20] C.B. Thorn, hep-th/0203167.
[21] J.S. Rozowsky and C.B. Thorn, Phys. Rev. D **60**, 045001 (1999).